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Coalgebra for Feynman Integrals

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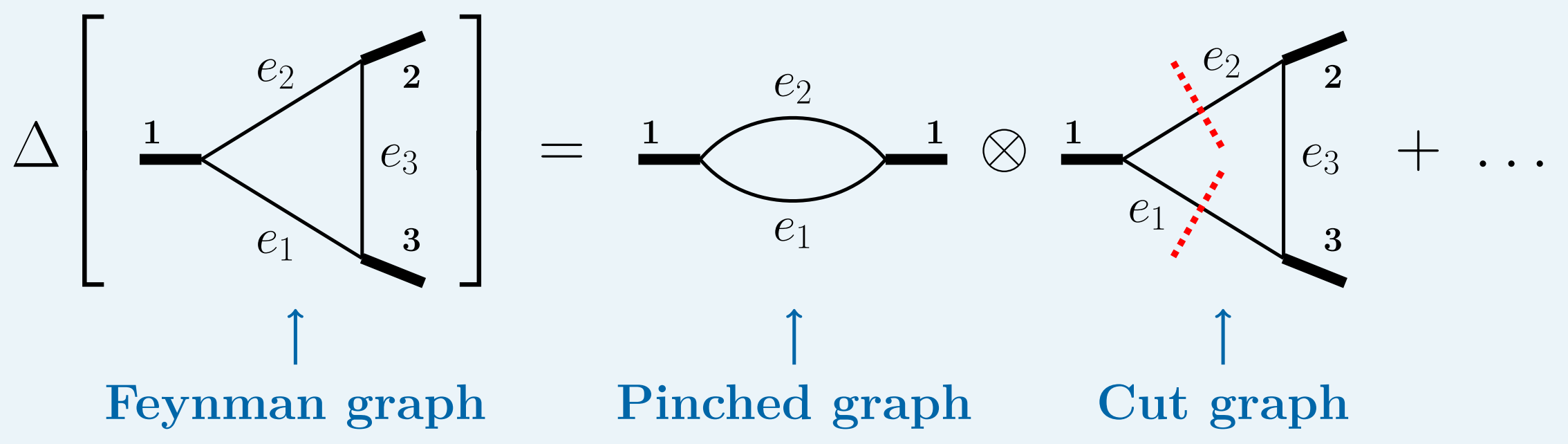
Introduction

Feynman integrals are the key components involved in precision calculations in quantum field theory, and are crucial to the computation of **scattering amplitudes**. However, they can often be difficult to evaluate. The search for more efficient computational techniques is likely to be aided by gaining a greater understanding of their **algebraic structure**.

In recent years, it has been conjectured that Feynman integrals obey a **coaction principle** [1], which postulates the existence of a mathematical operation called a *coaction* which allows Feynman integrals to be decomposed into pairs of simpler integrals.

This project focuses on the **diagrammatic coaction**, which realises the coaction in terms of operations performed on the corresponding Feynman graphs.



Multiple Polylogarithms and Coactions

Multiple polylogarithms (MPLs) form a class of functions that generalise the classical polylogarithms to several variables, and they arise in the computation of a large class of Feynman integrals [2].

MPLs have the **iterated integral representation**

$$G(z_1, z_2, \ldots, z_n; y) = \int_0^y \frac{dt}{t - z_1} G(z_2, \ldots, z_n; t).$$

In the special case where all z_i are equal to zero, we define

$$G(\vec{0}_n; y) = \frac{1}{n!} \log^n(y).$$

Let H be a unital associative algebra, and A be a $\mathbb{Q}$-vector space. A **coaction** is a linear map $\Delta : A \to A \otimes H$ which is

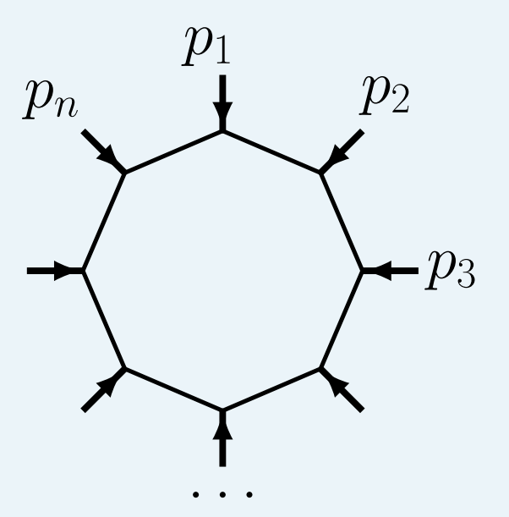
1. A homomorphism: $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b);$
2. Coassociative: $(\Delta \otimes \mathrm{id})\, \Delta = (\mathrm{id} \otimes \Delta)\, \Delta.$

Let $\mathcal{A}$ be the $\mathbb{Q}$-vector space spanned by all MPLs. This space can be endowed with a coaction $\Delta_{\text{MPL}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}/(i\pi\mathcal{A})$, where the second entry is only defined modulo $i\pi$. For the classical polylogarithms, this coaction is given by

$$\Delta_{\text{MPL}}(\mathrm{Li}_n(z)) = 1 \otimes \mathrm{Li}_n(z) + \sum_{k=0}^{n-1} \frac{1}{k!} \mathrm{Li}_{n-k}(z) \otimes \log^k(z).$$

One-Loop Feynman Integrals

We work in dimensional regularisation in $D = d - 2\epsilon$ dimensions. In the notation of [3], the scalar one-loop n-point Feynman integrals are defined as



$$\widetilde{J}_n = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \prod_{j=1}^{n} \frac{1}{(k - q_j)^2 - m_j^2}.$$

Loop momentum

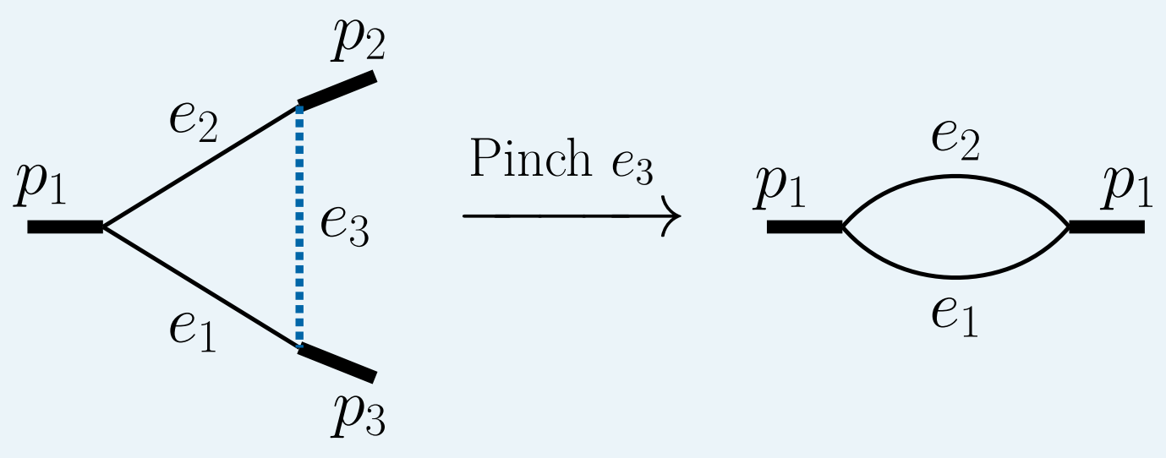
Linear combinations of external momenta

Masses

To discuss the diagrammatic coaction, we must introduce two different diagrammatic operations which can be performed on a one-loop graph.

Pinching a propagator

This corresponds to deleting a propagator and identifying the vertices at its endpoints, yielding a one-loop integral with fewer propagators.



These pinched graphs appear in the first entries of the diagrammatic coaction.

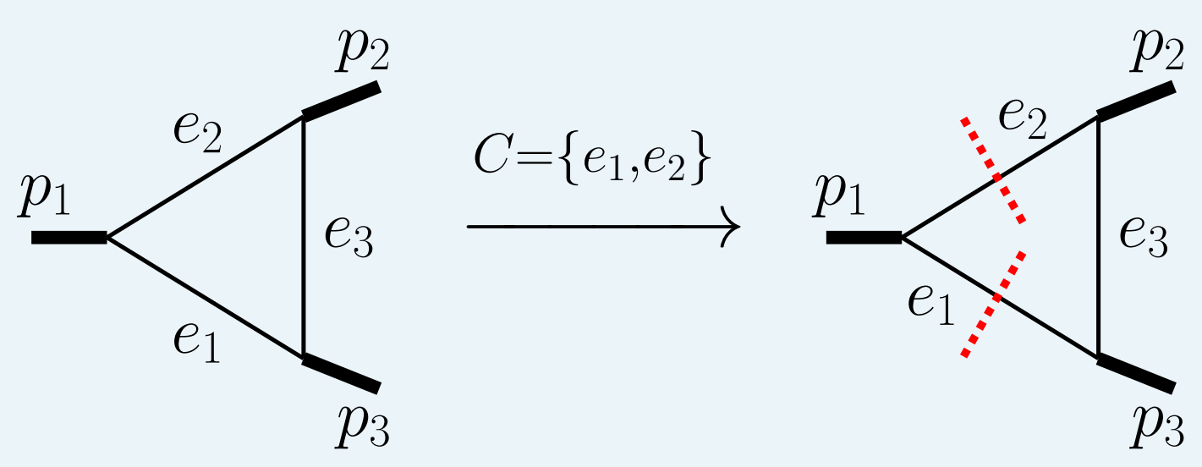
Notation:
Bold lines: Massive propagators
Light lines: Massless propagators

Cutting a propagator

This corresponds to replacing the propagator by a Dirac delta function which forces it to go on mass-shell.

$$\frac{1}{p^2 - m^2} \longrightarrow -2\pi i\delta(p^2 - m^2)$$

Letting C denote the subset of propagators which are cut, we obtain a cut Feynman integral $\mathcal{C}_C \widetilde{J}_n$.



The Diagrammatic Coaction at One Loop

To find each term in the diagrammatic coaction of a one-loop Feynman graph, start by choosing a nonempty subset C of propagators [3].

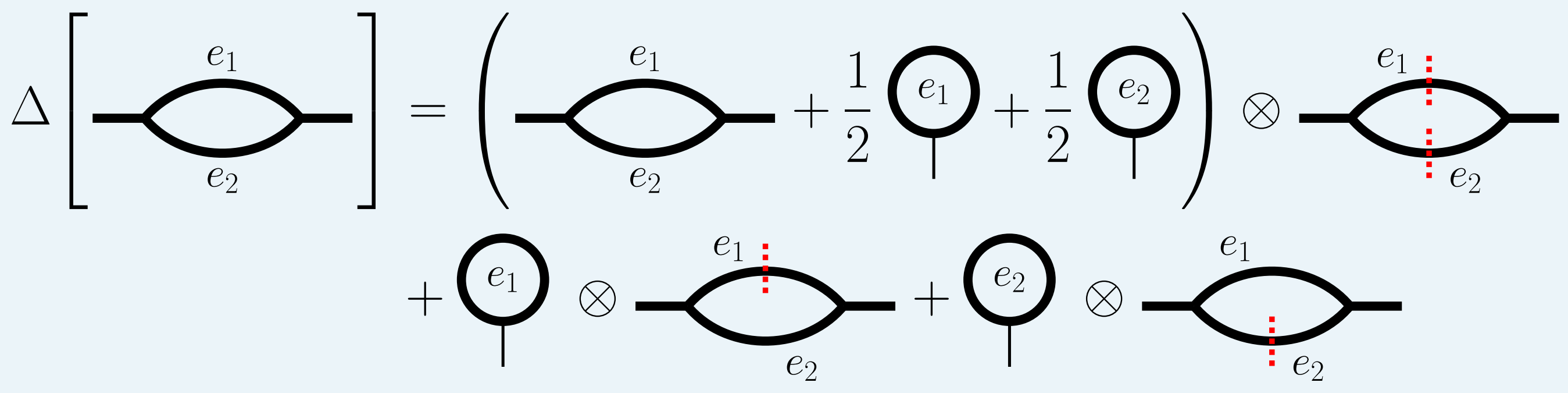
The **second entry** of that term is the cut Feynman integral $\mathcal{C}_C J_n$.

The corresponding **first entry** depends on the parity of $|C|$:

- $|C|$ odd: The first entry is the graph obtained by pinching the uncut edges.
- $|C|$ even: The first entry is the graph obtained by pinching the uncut edges, plus one-half times the sum of all graphs obtained by pinching an additional edge.

To obtain the full diagrammatic coaction, repeat this procedure for all possible nonempty subsets C and take the sum of the resulting terms.

Example: The diagrammatic coaction of the massive bubble integral is



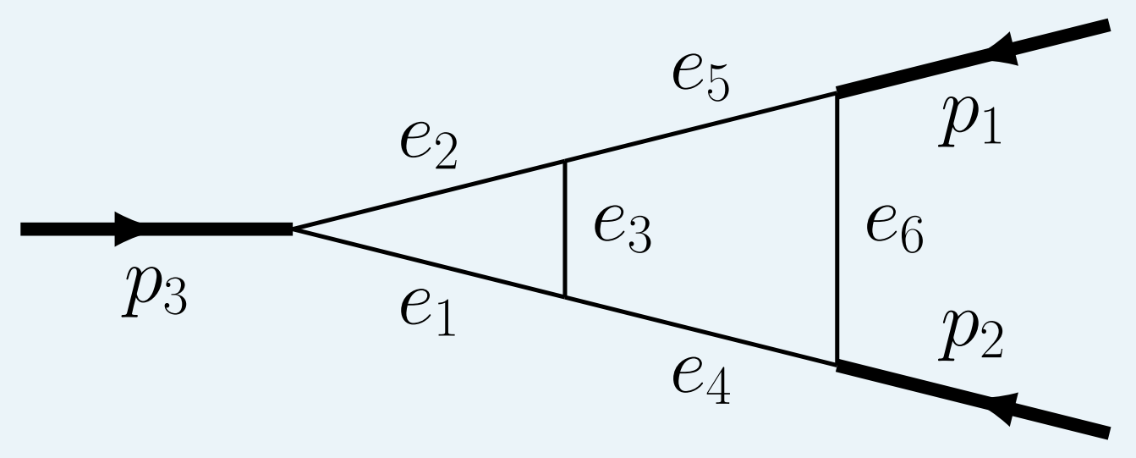
These diagrams can also be evaluated in terms of MPLs, and the diagrammatic coaction Δ agrees with the known coaction Δ_{MPL}.

The Diagrammatic Coaction Beyond One Loop

The diagrammatic coaction on one-loop Feynman integrals is well understood [2, 3], but there are difficulties associated with extending it to the multi-loop case. For example, multi-loop integrals may have

- More than one master integral with the same set of propagators.
- Several independent cut integrals which share the same set of on-shell propagators [4].

The main focus of this project has been the **two-loop three-point ladder diagram**:



$$z\bar{z} = \frac{p_2^2}{p_1^2}$$

$$(1 - z)(1 - \bar{z}) = \frac{p_3^2}{p_1^2}$$

In four dimensions, this integral evaluates to

$$T_L(p_1^2, p_2^2, p_3^2) = i(p_1^2)^{-2} \frac{1}{(1 - z)(1 - \bar{z})(z - \bar{z})} F(z, \bar{z}),$$

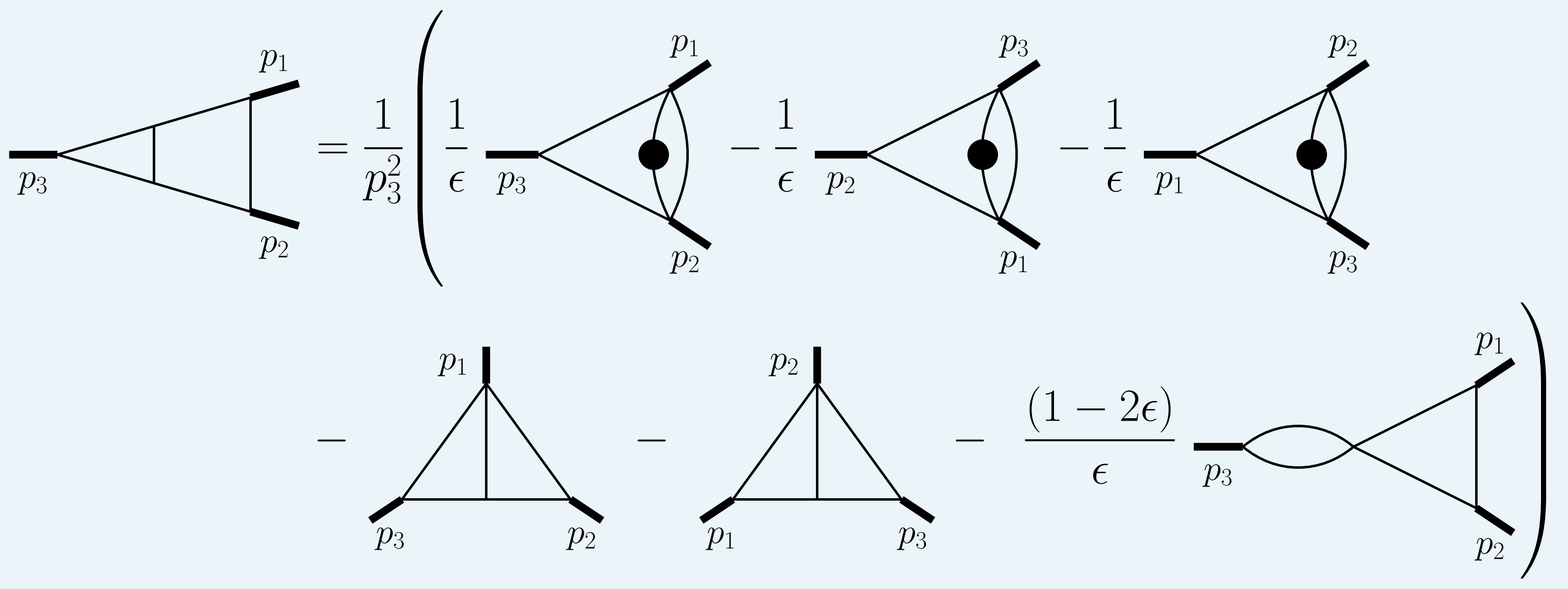
where

$$F(z, \bar{z}) = 6[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})] - 3\log(z\bar{z})[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})] + \frac{1}{2}\log^2(z\bar{z})[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})].$$

Since this is a function of MPLs, we can easily obtain the coaction Δ_{MPL}. To find a diagrammatic representation of this coaction, the tensor products must be arranged such that

- the **first entries** are expressed in terms of two-loop master integrals, and
- the **second entries** are expressed in terms of cuts of the original diagram.

The ladder can be reduced to its master integrals using the Mathematica package FIRE [5], which gives the reduction formula



These master integrals are expected to appear in the first entries of the diagrammatic coaction, while the second entries will be the corresponding cut integrals.

Summary

- The diagrammatic coaction allows Feynman graphs to be decomposed into **pairs of pinched and cut graphs**.
- It provides an insight into the **algebraic and analytic structure** of Feynman integrals.
- The diagrammatic coaction is well-understood at one loop, but extending it to the **multi-loop case** is still an area of ongoing research.

References

[1] F. Brown. "Feynman amplitudes, coaction principle, and cosmic Galois group". *Communications in Number Theory and Physics* 11 (2017), pp. 453–556. arXiv: `1512.06409 [math-ph]`.
[2] S. Abreu, R. Britto, C. Duhr, and E. Gardi. "Algebraic Structure of Cut Feynman Integrals and the Diagrammatic Coaction". *Physical Review Letters* 119 (5 2017), p. 051601. arXiv: `1703.05064 [hep-th]`.
[3] S. Abreu, R. Britto, C. Duhr, and E. Gardi. "Diagrammatic Hopf algebra of cut Feynman integrals: the one-loop case". *Journal of High Energy Physics* 2017.12 (2017). arXiv: `1704.07931v2 [hep-th]`.
[4] S. Abreu, R. Britto, C. Duhr, E. Gardi, and J. Matthew. "The diagrammatic coaction beyond one loop". *Journal of High Energy Physics* 2021.10 (2021). arXiv: `2106.01280 [hep-th]`.
[5] A. V. Smirnov and F. S. Chukharev. "FIRE6: Feynman Integral REduction with modular arithmetic". *Computer Physics Communications* 247 (2020), p. 106877. arXiv: `1901.07808 [hep-th]`.