Coalgebra for Feynman Integrals

Eliza Somerville

Supervisor: Professor Ruth Britto

20 February 2024

Motivation

Coaction Principle

There exists a coaction Δ which allows Feynman integrals to be decomposed into pairs of simpler integrals. (Brown, 2017)

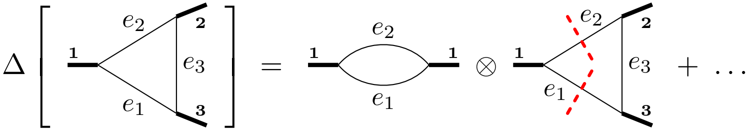
$$\Delta(J) = \sum_i J_i^{(1)} \otimes J_i^{(2)}$$

Feynman integral

Simpler integrals

Diagrammatic Coaction

Realised in terms of operations on Feynman graphs. (Abreu et al., 2017)



Feynman graph

Pinched graph

Cut graph

Multiple Polylogarithms (MPLs)

Iterated integral representation of MPLs:

$$G(a_1, a_2, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t).$$

In the special case where all a_i are equal to zero,

$$G(\vec{0}_n; z) = \frac{1}{n!} \log^n(z).$$

Examples:

$$G(0; z) = \log z, \qquad G(a; z) = \log\left(1 - \frac{z}{a}\right),$$

$$G(\vec{a}_n; z) = \frac{1}{n!} \log^n\left(1 - \frac{z}{a}\right), \quad G(\vec{0}_{n-1}, a; z) = -\operatorname{Li}_n\left(\frac{z}{a}\right).$$

The Coaction on Multiple Polylogarithms

Let $\mathcal{A}$ be the $\mathbb{Q}$-vector space spanned by all MPLs.

There exists a coaction $\Delta_{\text{MPL}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}/(i\pi\mathcal{A})$, which is

1. A homomorphism:
$$\Delta_{\text{MPL}}(a \cdot b) = \Delta_{\text{MPL}}(a) \cdot \Delta_{\text{MPL}}(b)$$
2. Coassociative: $(\Delta_{\text{MPL}} \otimes \text{id})\Delta_{\text{MPL}} = (\text{id} \otimes \Delta_{\text{MPL}})\Delta_{\text{MPL}}$

For the classical polylogarithms,

$$\Delta_{\text{MPL}}\left(\text{Li}_n(z)\right) = 1 \otimes \text{Li}_n(z) + \sum_{k=0}^{n-1} \frac{1}{k!} \text{Li}_{n-k}(z) \otimes \log^k(z)$$

Example: $\Delta_{\text{MPL}}\left(\text{Li}_2(z)\right) = 1 \otimes \text{Li}_2(z) - \log(1-z) \otimes \log(z) + \text{Li}_2(z) \otimes 1$

One-Loop Feynman Integrals

We work in dimensional regularisation in $D = d - 2\epsilon$ dimensions.

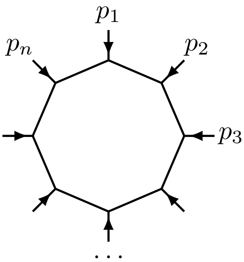
The scalar one-loop n-point Feynman integrals are defined as

Loop momentum

$$\widetilde{J}_n = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \prod_{j=1}^{n} \frac{1}{(k-q_j)^2 - m_j^2}.$$

Linear combinations
of external momenta

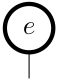
Masses



Examples

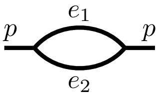
Tadpole integral (one-point):

Bold: Massive
Light: Massless



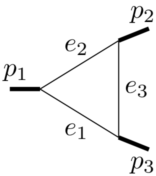
$$= \widetilde{J}_1(m^2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 - m^2}$$

Bubble integral (two-point):



$$= \widetilde{J}_2(p^2; m_1^2, m_2^2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 - m_1^2)((k-p)^2 - m_2^2)}$$

Triangle integral (three-point):

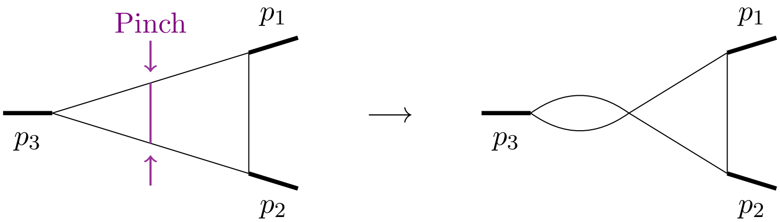


$$= \widetilde{J}_3(p_1^2, p_2^2, p_3^3) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$

Pinches and Master Integrals

First entries in coaction = pinches of original graph.

Pinch a propagator → Delete it and identify its endpoints.



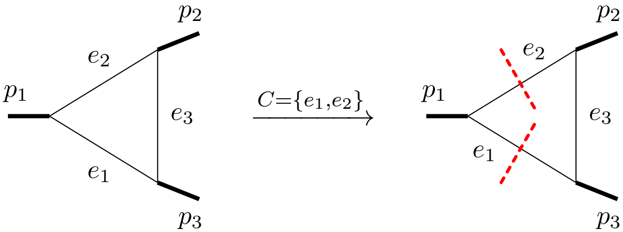
Related to master integrals: Basis elements of the topology defined by the graph.

Cut Feynman Integrals

Second entries in coaction = cuts of original graph.

$$\text{Feynman integral } \widetilde{J}_n \longrightarrow \text{Cut Feynman integral } \mathcal{C}_C\widetilde{J}_n$$

Set of cut propagators

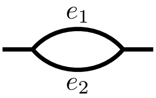


Cut propagators are replaced according to

$$\frac{1}{p^2-m^2} \longrightarrow -2\pi i\delta(p^2-m^2)$$

The Diagrammatic Coaction at One Loop

Rule for computing diagrammatic coaction of J_n (Abreu et al., 2017)



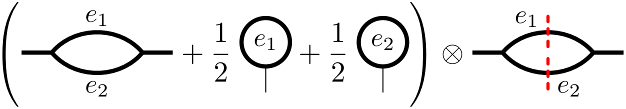
Each term: Choose a nonempty subset C of propagators.

Second entry = Cut Feynman integral $\mathcal{C}_C J_n$.

First entry:

- $|C|$ odd: First entry is the graph obtained by pinching the uncut edges.

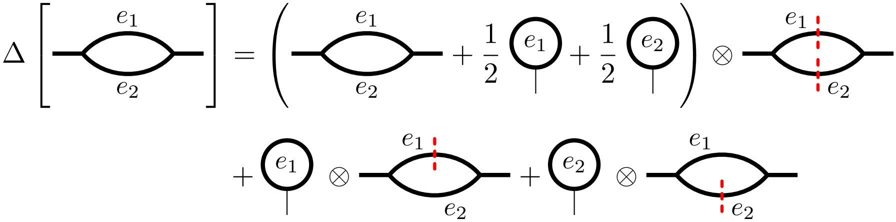
- $|C|$ even: First entry is the graph obtained by pinching the uncut edges, plus one-half times the sum of all graphs obtained by pinching an additional edge.



Repeat for all possible nonempty subsets C and take sum.

The Massive Bubble Integral

Full diagrammatic coaction of the massive bubble integral:



These diagrams can also be evaluated in terms of MPLs.

The diagrammatic coaction Δ agrees with the known coaction Δ_{MPL}.

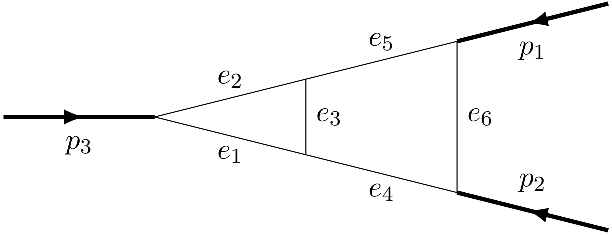
Beyond One Loop

The diagrammatic coaction on one-loop integrals is well-understood.
It has yet to be fully generalised to higher-loop integrals.

Difficulties that arise beyond one loop: (Abreu et al., 2021)

- There may be more than one master integral with the same set of propagators.
- There may be several independent cut integrals which share the same set of on-shell propagators.

Example: The Two-Loop Three-Point Ladder



$$z\bar{z} = \frac{p_2^2}{p_1^2}$$

$$(1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}$$

In four dimensions:

$$T_L(p_1^2, p_2^2, p_3^2) = i(p_1^2)^{-2} \frac{1}{(1-z)(1-\bar{z})(z-\bar{z})} F(z,\bar{z}),$$

where

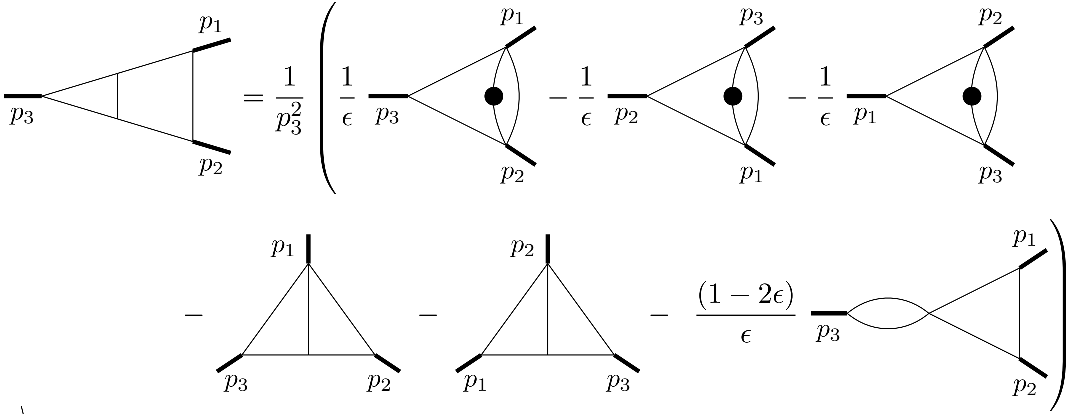
$$F(z,\bar{z}) = 6\left[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})\right] - 3\log(z\bar{z})\left[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})\right] + \frac{1}{2}\log^2(z\bar{z})\left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\right].$$

Coaction on the Two-Loop Three-Point Ladder

$$\begin{aligned}
\Delta_{\mathrm{MPL}}[F(z,\bar{z})] = {} & F(z,\bar{z}) \otimes 1 + 1 \otimes F(z,\bar{z}) \\
& + \log(z\bar{z}) \otimes \left[-3\,\mathrm{Li}_3(z) + 3\,\mathrm{Li}_3(\bar{z}) + \log(z\bar{z})(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}))\right] \\
& + \log((1-z)(1-\bar{z})) \otimes \frac{1}{2}\log z \log \bar{z} \log \frac{z}{\bar{z}} \\
& + \left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \log(1-z)\log(z\bar{z})\right] \otimes \frac{1}{2}\left(\log^2 z - 2\log z \log \bar{z}\right) \\
& + \left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1-\bar{z})\log(z\bar{z})\right] \otimes \frac{1}{2}\left(\log^2 \bar{z} - 2\log z \log \bar{z}\right) \\
& + \frac{1}{2}\log^2(z\bar{z}) \otimes (\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})) \\
& + \left[3\,\mathrm{Li}_3(z) + 3\,\mathrm{Li}_3(\bar{z}) - \log(z\bar{z})(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}))\right] \otimes \log \frac{z}{\bar{z}} \\
& - \frac{1}{2}\log(z\bar{z})\left[2\,\mathrm{Li}_2(z) + \log(1-z)\log(z\bar{z})\right] \otimes \log z \\
& + \frac{1}{2}\log(z\bar{z})\left[2\,\mathrm{Li}_2(\bar{z}) + \log(1-\bar{z})\log(z\bar{z})\right] \otimes \log \bar{z}
\end{aligned}$$

Reduction to Master Integrals

Using the C++ version of FIRE (Feynman Integral REduction): (Smirnov and Chukharev, 2020)

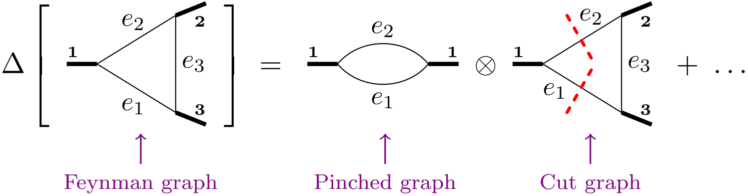



= Squared propagator

Conclusion

In summary:

- For one-loop integrals, there is a well-understood diagrammatic form of the coaction.



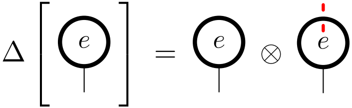
- The next step in the study of the diagrammatic coaction is to generalise it to multi-loop integrals.

References

- Abreu, Samuel, Ruth Britto, Claude Duhr and Einan Gardi (July 2017). 'Algebraic Structure of Cut Feynman Integrals and the Diagrammatic Coaction'. *Phys. Rev. Lett.* 119 (5), p. 051601. arXiv: `1703.05064 [hep-th]`.
- — (Dec. 2017). 'Diagrammatic Hopf algebra of cut Feynman integrals: the one-loop case'. *Journal of High Energy Physics* 2017.12. arXiv: `1704.07931v2 [hep-th]`.
- Abreu, Samuel, Ruth Britto, Claude Duhr, Einan Gardi and James Matthew (Oct. 2021). 'The diagrammatic coaction beyond one loop'. *Journal of High Energy Physics* 2021.10. arXiv: `2106.01280 [hep-th]`.
- Brown, Francis (2017). 'Feynman amplitudes, coaction principle, and cosmic Galois group'. *Commun. Num. Theor. Phys.* 11, pp. 453–556. arXiv: `1512.06409 [math-ph]`.
- Smirnov, A.V. and F.S. Chukharev (Feb. 2020). 'FIRE6: Feynman Integral REduction with modular arithmetic'. *Computer Physics Communications* 247, p. 106877. arXiv: `1901.07808 [hep-th]`.

Other Examples

Diagrammatic coaction of the tadpole integral:



Diagrammatic coaction of the triangle integral: