Modularity of Black Holes and Ramanujan's Tau Function

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Outline

1. Introduction
2. Partition Functions in String Theory
3. Elliptic Genus of K3
4. Counting Black Hole Microstates
5. On the Ramanujan τ Function

Title slide: Contour plot of the modular form $(2\pi)^{12}\eta^{24}(\tau)$, plotted in Python using `cplot` [20].

Background: Gravity

In 1687, Newton published his first theory of gravity, with his universal gravitational law of attraction for two masses:

$$\underbrace{F}_{\text{Force}} = \frac{\overbrace{G}^{\text{Newton's gravitational constant}}\overbrace{Mm}^{\text{Masses}}}{\underbrace{r^2}_{\text{Distance between masses}}}.$$

Although powerful, it failed to explain known phenomena.

In 1915, Albert Einstein published a new theory of gravity, *general relativity* (GR), and introduced the concept of *spacetime*.

GR has been able to explain phenomena that Newton could not, however there are still inconsistencies.

Background: Quantum Physics

In 1900, Max Planck introduced the idea of packets of energy, *quanta*, to explain blackbody radiation. This would become the foundation to the theory of *quantum mechanics* (QM), the study of subatomic particles.

One of the most important equations in QM is the Schrödinger equation, which describes the position and momentum of a wavefunction ψ:

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + \overset{\text{Potential}}{V}(\mathbf{r})\psi(\mathbf{r},t).$$

This equation does not however account for relativistic effects, or apply in scenarios with high energies.

Background: Quantum Physics

In 1928, Paul Dirac derived a relativistic version of the Schrödinger equation:

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0.$$

This equation is at the forefront of *quantum field theory* (QFT), and accounts for the disparities of the Schrödinger equation. QFT better explains subatomic interactions.

Examples of QFTs include quantum electrodynamics (QED), quantum chromodynamics (QCD), and the standard model.

Question:

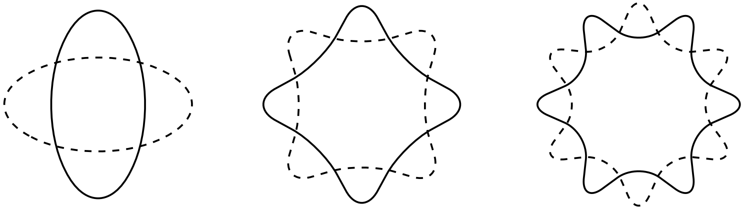
Can we construct a theory which combines both GR and QFT?

What is String Theory?

A major aim of modern physics is to unify GR and QFT into a single fundamental theory that fully describes all of the laws of nature.

One of the prime candidates for this unifying theory is *string theory*, where point particles are replaced by one-dimensional strings.

Particles arise as oscillator modes of these strings.



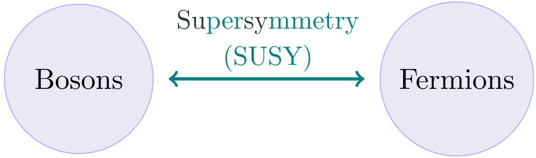
Based on an image from Becker et al.[1]

[1] Katrin Becker, Melanie Becker, and John H. Schwarz. *String Theory and M-Theory: A Modern Introduction*. Cambridge University Press, 2006.

Supersymmetry

The string theories we will be working with require *supersymmetry*.

Supersymmetry is a symmetry which relates bosons to fermions.



BPS state: A state which preserves some subset of the full set of supersymmetries of the theory.

We will consider 1/4 BPS states, which preserve one quarter of the original supersymmetries.[2]

[2] Joseph Gerard Polchinski. *String Theory, Volume I: An Introduction to the Bosonic String.* Cambridge University Press, 1998.

What are Black Holes?

In 1916, Karl Schwarzschild found a spherically symmetric solution to Einstein's equation in vacuum, given by the line element

Newton's gravitational constant

Mass of the black hole

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Speed of light

where we use spherical coordinates (r, θ, ϕ).

We can see that the metric is singular at $r = 0$ and $r = 2GM/c^2$, with the latter being called the *Schwarzschild radius* r_s.

The surface $r = r_s$ forms a boundary called the *event horizon*, and the enclosed region within the event horizon is called a *black hole*.[3]

[3] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* Benjamin Cummings, 2004.

Charged Black Holes

The most general static, spherically symmetric, *charged* solution of Einstein's field equation is the Reissner–Nordström metric, given by[4]

$$ds^2 = -\frac{\Delta(r)}{r^2}\,dt^2 + \left(\frac{\Delta(r)}{r^2}\right)^{-1} dr^2 + r^2\,d\Omega^2,$$

where

Electric charge Magnetic charge

$$\Delta(r) = r^2 - 2Mr + Q^2 + P^2.$$

Of interest to us are *extremal* black holes, where $M = \sqrt{Q^2 + P^2}$.

We will be considering string-theoretic extremal black holes, where the charges Q and P are vectors.

Note: Here and on all subsequent slides, we use natural units, in which $c = \hbar = k_{\mathrm{B}} = G = 1$.

[4] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* Benjamin Cummings, 2004.

Black Hole Entropy

Hawking showed that the inclusion of quantum effects allows black holes to radiate.[5]

A semiclassical treatment predicts that the entropy S of a black hole is related to its area A by

$$S_{\mathrm{BH}} = \frac{A}{4}.$$

This is the *Bekenstein–Hawking entropy*.

In a consistent theory of quantum gravity, we should be able to write this as a logarithm of the microstate degeneracy d:

Magnetic charge

$$S_{\mathrm{stat}} = \ln d(Q, P).$$

Electric charge

[5] Stephen W. Hawking. "Particle creation by black holes". *Communications in Mathematical Physics* 43.3 (1975), pp. 199–220.

Black Hole Entropy: A Test of String Theory

Bekenstein–Hawking entropy

$$S_{\mathrm{BH}} = \frac{A}{4}$$

Macroscopic

Computed using black hole area

Black Hole Entropy

Computed using microstate degeneracy[6]

Statistical entropy

$$S_{\mathrm{stat}} = \ln d(Q, P)$$

Microscopic

[6] Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde. "Counting dyons in $N = 4$ string theory". *Nuclear Physics B* 484.3 (Jan. 1997), pp. 543–561. arXiv: `hep-th/9607026 [hep-th]`.

Why Modular Forms?

The study of modular forms themselves is an exercise in complex analysis. Modular forms are simply complex functions with some strange transformation properties. Despite this, they have seen remarkable applications in various seemingly unconnected areas including

1. Number Theory[7]
2. Group Theory[8]
3. String Theoretic Black Holes

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[7] Andrew Wiles. "Modular Elliptic Curves and Fermat's Last Theorem". *Annals of Mathematics* 141.3 (1995), pp. 443–551.

[8] Valdo Tatitscheff. "A short introduction to Monstrous Moonshine". (2019). arXiv: 1902.03118 [math.NT].

What are Modular Forms?

We first discuss some ideas we will need to define modular forms.

1. The Upper Half Plane, $\mathbb{H}$, is defined as subset of $\mathbb{C}$ with strictly positive imaginary part. That is $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.
2. The Special Linear Group $\mathrm{SL}_2(\mathbb{Z})$ is the group of all 2×2 matrices with integer coefficients and determinant 1.

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

What are Modular Forms?

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $k \in \mathbb{Z}$ if it is bounded as $\tau \to i\infty$ and

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.[9]

[9] Tom M. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. 2nd ed. Graduate Texts in Mathematics 41. Springer New York, 1976.

Transformations of Modular Forms

We will now look at two key transformations of modular forms under the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, recalling that modular forms transform as

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$f\left(\frac{\tau + 1}{0 + 1}\right) = (0 + 1)^k f(\tau)$$

$$\Rightarrow f(\tau + 1) = f(\tau).$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$f\left(\frac{0 - 1}{\tau + 0}\right) = (\tau + 0)^k f(\tau)$$

$$\Rightarrow f\left(\frac{-1}{\tau}\right) = \tau^k f(\tau).$$

Dedekind η Function

The Dedekind η function is a modular form of weight $\frac{1}{2}$ given by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

where $q = e^{2\pi i \tau}$.
Under the modular transformations $\tau \to \tau + 1$ and $\tau \to \frac{-1}{\tau}$ the η function transforms as

$$\eta(\tau + 1) = e^{(2\pi i)/24} \eta(\tau),$$
$$\eta\left(\frac{-1}{\tau}\right) = (-i\tau)^{1/2} \eta(\tau).$$

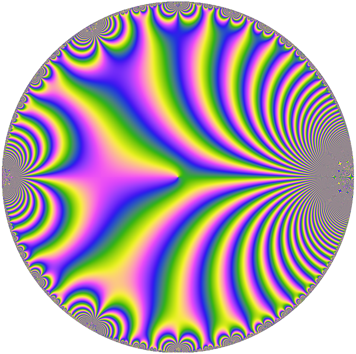


Figure: A phase plot of the modular discriminant $\Delta = \eta^{24}$.

Superstring Theory

A *superstring theory* is a string theory which incorporates supersymmetry between bosons and fermions.

Bosonic String

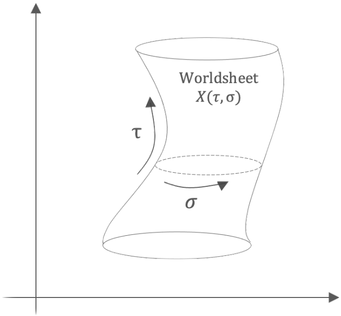
Fermionic String

Superstring Theory

(supersymmetric string theory)

Only consistent in spacetime of dimension $D = 10$

The Bosonic String



Based on an image from Tong.[10]

Figure: The bosonic string: A 1-D object that sweeps out a 2-D surface called the worldsheet, described by $X(\tau, \sigma)$.

Classically, the Lagrangian of the closed bosonic string is

$$L = \frac{1}{4\pi\alpha'} \int_0^{2\pi} (\dot{X}^2 - X'^2)\, d\sigma,$$

with corresponding Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi\alpha'} (\dot{X}^2 - X'^2).$$

The momentum density is thus

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{1}{2\pi\alpha'} \dot{X}.$$

[10] David Tong. "Lectures on String Theory". (2009). arXiv: 0908.0333 [hep-th].

The Bosonic String

Applying the Euler-Lagrange equation

$$\partial_\tau\left(\frac{\partial\mathcal{L}}{\partial_\tau X}\right)+\partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial_\sigma X}\right)-\frac{\partial\mathcal{L}}{\partial X}=0$$

gives the resulting equation of motion:

$$\ddot{X}-X''=0.$$

The general solution may be represented using oscillator modes as

Convenient normalisation

Left-moving modes

Right-moving modes

$$X(\tau,\sigma)=i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{\alpha_k}{k}e^{-ik\tau+ik\sigma}+i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{\tilde{\alpha}_k}{k}e^{-k\tau-ik\sigma}+\underbrace{\alpha' p\tau+x}_{\text{Zero modes}}.$$

The left-moving and right-moving solutions correspond to the functions $\tau-\sigma$ and $\tau+\sigma$ respectively.

Quantisation of the Bosonic String

The classical Hamiltonian for the bosonic string is

$$\mathcal{H} = \int_0^{2\pi} (\Pi \dot{X} - \mathcal{L})\, d\sigma = \frac{1}{4\pi\alpha'} \int_0^{2\pi} (\dot{X}^2 + X'^2)\, d\sigma \,.$$

We first consider a closed bosonic string with periodic boundary conditions, $X(\tau, \sigma) = +X(\tau, \sigma + 2\pi)$ (and neglect zero modes).

To quantise the system, we impose the commutation relation

$$\left[X(\sigma), \Pi(\sigma')\right] = i\delta(\sigma - \sigma'),$$

which ultimately gives[11]

$$\mathcal{H}_L = -\frac{1}{24} + \sum_{k=1}^{\infty} \alpha_{-k}\alpha_k, \qquad \mathcal{H}_R = -\frac{1}{24} + \sum_{k=1}^{\infty} \tilde{\alpha}_{-k}\tilde{\alpha}_k.$$

[11] Here, we used zeta function regularisation to obtain $\sum_{k=1}^{\infty} k = -\frac{1}{12}$.

The **partition function** Z is a quantity from statistical mechanics which relates the microscopic configurations of a system to its macroscopic variables.

The general form of the partition function is given by the trace[12]

Hamiltonian for left-movers

Hamiltonian for right-movers

$$Z = \operatorname{tr}\left[e^{-\beta_L \mathcal{H}_L - \beta_R \mathcal{H}_R}\right]$$

$1 + e^{-\beta_L k} + e^{-2\beta_L k} + \dots$

$$= e^{\frac{\beta_L}{24}} \prod_{k=1}^{\infty} \operatorname{tr}\left(e^{-\beta_L \alpha_{-k}\alpha_k}\right) e^{\frac{\beta_R}{24}} \prod_{k=1}^{\infty} \operatorname{tr}\left(e^{-\beta_R \tilde{\alpha}_{-k}\tilde{\alpha}_k}\right)$$

$$= e^{\frac{\beta_L}{24}} \prod_{k=1}^{\infty} \frac{1}{1 - e^{-\beta_L k}} e^{\frac{\beta_R}{24}} \prod_{k=1}^{\infty} \frac{1}{1 - e^{-\beta_R k}}.$$

Thermodynamic beta; $\beta_L = -2\pi i \tau$, $\beta_R = 2\pi i \bar{\tau}$

[12] Joseph Gerard Polchinski. *String Theory, Volume I: An Introduction to the Bosonic String.* Cambridge University Press, 1998.

Bosonic Partition Functions

Given that $q = e^{2\pi i\tau}$ and $\bar{q} = e^{-2\pi i\bar{\tau}}$:

$$
\begin{aligned}
Z &= \frac{1}{q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1-q^k)} \frac{1}{\bar{q}^{\frac{1}{24}} \prod_{k=1}^{\infty}(1-\bar{q}^k)} \\
&= \frac{1}{\eta(q)} \frac{1}{\eta(\bar{q})} = \frac{1}{|\eta(q)|^2}.
\end{aligned}
$$

We will later introduce a $\mathbb{Z}_2$ orbifold, on which we can also consider anti-periodic boundary conditions on the closed string: $X(\tau,\sigma) = -X(\tau,\sigma+2\pi)$.

The total partition function in that case is given by:

$$
Z = e^{-\frac{\beta_L}{48}} \prod_{k=1}^{\infty} \frac{1}{1-e^{-\beta_L(k-1/2)}} e^{-\frac{\beta_R}{48}} \prod_{k=1}^{\infty} \frac{1}{1-e^{-\beta_R(k-1/2)}}.
$$

The Fermionic String

The fermionic case follows from the same general methodology as the bosonic case.

Bosonic String $X(\tau, \sigma)$	$\leftrightarrow$	Fermionic String $\psi(\tau, \sigma)$

Left-moving modes $\longleftarrow$ α_k

$\tilde{\alpha}_k$ $\longrightarrow$ Right-moving modes

Left-moving modes $\longleftarrow$ ψ_k

$\tilde{\psi}_k$ $\longrightarrow$ Right-moving modes

Commutation relations:
$[X(\sigma), \Pi(\sigma')] \propto i\delta(\sigma - \sigma')$

Anticommutation relations:
$\{\psi(\sigma), \Pi(\sigma')\} \propto i\delta(\sigma - \sigma')$

Fermionic Witten Index

The *Witten index* is a generalisation of the partition function, defined by

$$\chi = \mathrm{tr}\left[(-1)^{F_L+F_R} e^{-\beta_L \mathcal{H}_L - \beta_R \mathcal{H}_R}\right].$$

Fermion number operator

The periodic fermionic Witten index (without zero modes) is given by:

$$\chi = \eta(q)\eta(\bar{q}) = |\eta(q)|^2.$$

The anti-periodic fermionic Witten index is given by:

$$\chi = e^{\frac{\beta_L}{48}} \prod_{k=1}^{\infty}(1 - e^{-\beta_L(k-1/2)}) e^{\frac{\beta_R}{48}} \prod_{k=1}^{\infty}(1 - e^{-\beta_R(k-1/2)}).$$

$Z_{\mathrm{K3}}(\tau, z) \longleftarrow$ Elliptic genus of K3

Fourier coefficients $\big\downarrow$ $c(n, r)$

$\dfrac{1}{\tilde{\Phi}_{10}(\rho, \sigma, v)} \longleftarrow$ Igusa cusp form of weight 10

Fourier coefficients $\big\downarrow$ $g(m, n, p)$

$d(Q, P) \longleftarrow$ Black hole microstate degeneracy

Compactification

Superstring theories are typically formulated in ten-dimensional Minkowski spacetime $\mathcal{M}_{10}$.

To reconcile this with our everyday experience, we need a *compactification* to four dimensions:

$$\mathcal{M}_{10} = \mathbb{R}^{1,3} \times X_6.$$

4D Minkowski spacetime

Target space
(6D manifold)

We will consider the case of type II string theory compactified on $X_6 = \mathrm{K3} \times T^2$.[13]

[13] Atish Dabholkar and Suresh Nampuri. "Quantum Black Holes". *Strings and Fundamental Physics.* Springer Berlin Heidelberg, 2012, pp. 165–232. arXiv: 1208.4814 [hep-th].

A *K3 surface* is a type of complex manifold of dimension two (real dimension four) which is often used in string theory compactifications.

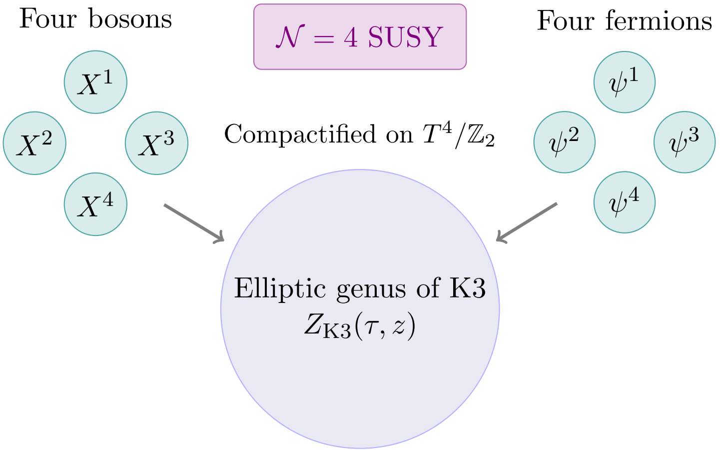
To count black hole microstates on $\mathrm{K3} \times T^2$, we need to compute a topological invariant of K3 called the *elliptic genus*.

We can do this by realising K3 as a superconformal field theory (SCFT) in the orbifold limit $T^4/\mathbb{Z}_2$.[14,15]

[14] Katrin Wendland. "K3 en route: From Geometry to Conformal Field Theory". (2015). arXiv: 1503.08426 [math.DG].

[15] Tohru Eguchi et al. "Superconformal algebras and string compactification on manifolds with $SU(n)$ holonomy". *Nuclear Physics B* 315.1 (1989), pp. 193–221.

Elliptic Genus of K3



Group action of $\mathbb{Z}_2$:

$$g : (X^\mu, \psi^\mu) \to (-X^\mu, -\psi^\mu)$$

Elliptic Genus of K3

The elliptic genus is defined[16] to be the following trace, taken over the Ramond-Ramond sector:

Angular momentum; $J_{\mathrm{K3}} = \pm 1$

$$Z_{\mathrm{K3}}(\tau, z) = \frac{1}{2} \sum_{a,b=0}^{1} \mathrm{tr}_{\mathrm{RR}g^a} \left[(-1)^{F_{\mathrm{K3}} + \bar{F}_{\mathrm{K3}}} g^b y^{J_{\mathrm{K3}}} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right],$$

Fermion number operator

Central charge; 6 for K3

where $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$.

This seems to depend on $\bar{q}$ (and hence $\bar{\tau}$) as well as q, but due to supersymmetry this dependence cancels.

[16] Edward Witten. "On the Landau-Ginzburg Description of $N = 2$ Minimal Models". *International Journal of Modern Physics A* 09.27 (Oct. 1994), pp. 4783–4800. arXiv: `hep-th/9304026 [hep-th]`.

Elliptic Genus of K3: $(a, b) = (1, 0)$

Twisted sector $\Rightarrow$ Anti-periodic boundary conditions

- Number of states in the twisted sector: $2^4 = 16$
- Fermionic modes:

$$\prod_{k=1}^{\infty} \left(1 - q^{k-1/2} y\right)^2 \left(1 - q^{k-1/2} y^{-1}\right)^2 \left(1 - \bar{q}^{k-1/2}\right)^4$$

- Bosonic modes:

$$\prod_{k=1}^{\infty} \frac{1}{\left(1 - q^{k-1/2}\right)^4} \frac{1}{\left(1 - \bar{q}^{k-1/2}\right)^4}$$

Total:

$$8 \prod_{k=1}^{\infty} \frac{\left(1 - q^{k-1/2} y\right)^2 \left(1 - q^{k-1/2} y^{-1}\right)^2}{\left(1 - q^{k-1/2}\right)^4} = 8 \frac{\theta_4^2 (\tau, z)}{\theta_4^2 (\tau, 0)}$$

Elliptic Genus of K3: Full Result

$(a, b) = (1, 1)$:

$$8 \prod_{k=1}^{\infty} \frac{\left(1 + q^{k-1/2} y\right)^2 \left(1 + q^{k-1/2} y^{-1}\right)^2}{\left(1 + q^{k-1/2}\right)^4} = 8 \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)}$$

$(a, b) = (0, 1)$:

$$8 \cos^2(\pi z) \prod_{k=1}^{\infty} \frac{\left(1 + q^{k} y\right)^2 \left(1 + q^{k} y^{-1}\right)^2}{\left(1 + q^{k}\right)^4} = 8 \frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)}$$

Hence the elliptic genus of K3 is

$$Z_{\mathrm{K3}}(\tau, z) = 8 \left(\frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)} + \frac{\theta_4^2(\tau, z)}{\theta_4^2(\tau, 0)} \right).$$

This is a *Jacobi form* of index one and weight zero.

Jacobi Forms

A Jacobi form of weight k and index m is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying:

1. $\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \exp\left(\frac{2\pi imcz}{c\tau+d}\right)\phi(\tau, z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
2. $\phi(\tau, z + \lambda\tau + \mu) = \exp\left(-2\pi im\left(\lambda^2\tau + 2\lambda z\right)\right)\phi(\tau, z)$ for all $\lambda, \mu \in \mathbb{Z}$.
3. ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{r^2 \leq 4n} c(n, r)\, q^n y^r,$$

where $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$.[17]

[17] Martin Eichler and Don Zagier. *The Theory of Jacobi Forms*. Progress in Mathematics 55. Birkhäuser Basel, 1985.

Jacobi Theta Functions

An important example of Jacobi forms are the Jacobi theta functions which appear in the equation for the elliptic genus. They are defined as follows:

$$\theta_1(\tau, z) = -i \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{\pi i (2n+1) z}$$

$$\theta_2(\tau, z) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{\pi i (2n+1) z}$$

$$\theta_3(\tau, z) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n z}$$

$$\theta_4(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n z}.$$

Jacobi Theta Functions

It is important to note that in the derivation of the elliptic genus, these theta functions appear, not in the sum form given on the previous slide but in their product form.

For example, for θ_3 we have

$$\theta_3(\tau, z) = \prod_{m=1}^{\infty} (1 - q^m)\left(1 + q^{\frac{2m-1}{2}} y\right)\left(1 + \frac{q^{\frac{2m-1}{2}}}{y}\right),$$

where as before, $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$.

Igusa Cusp Form of Weight Ten

The double expansion of the elliptic genus of K3 is

$$Z_{\mathrm{K3}}(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n,r) q^n y^r .$$

This can be used to find the multiplicative lift of the elliptic genus:

$$\tilde{\Phi}_{10}\left(\rho, \sigma, v\right) = e^{2\pi i(\rho+\sigma+v)} \prod_{\substack{k,l,j\in\mathbb{Z}, k,l\geq 0 \\ j<0 \text{ for } k=l=0}} \left(1 - e^{2\pi i(k\sigma+l\rho+jv)}\right)^{c(kl,j)} .$$

This is the Igusa cusp form of weight ten.[18]

[18] Ashoke Sen. "Black hole entropy function, attractors and precision counting of microstates". *General Relativity and Gravitation* 40.11 (Apr. 2008), pp. 2249–2431. arXiv: 0708.1270 [hep-th].

The degeneracy of black hole microstates is

$$d(Q,P) = (-1)^{Q\cdot P+1} \int_{\mathcal{C}} e^{-2\pi i\left(\rho Q^2/2+\sigma P^2/2+v Q\cdot P\right)} \frac{1}{\tilde{\Phi}_{10}\left(\rho,\sigma,v\right)}\, d\rho\, d\sigma\, dv\,.$$

Hence if we make the Fourier series expansion

$$\frac{1}{\tilde{\Phi}_{10}\left(\rho,\sigma,v\right)} = \sum_{m,n,p} g(m,n,p) e^{2\pi i(m\rho+n\sigma+pv)},$$

then

$$d(Q,P) = (-1)^{Q\cdot P+1} g\left(\frac{Q^2}{2}, \frac{P^2}{2}, Q\cdot P\right).$$

The contour $\mathcal{C}$ corresponds to the expansion of $1/\tilde{\Phi}_{10}$ first in powers of $e^{2\pi i\rho}, e^{2\pi i\sigma}$, and then in powers of $e^{-2\pi i v}$.[19]

[19] Ashoke Sen. “Black hole entropy function, attractors and precision counting of microstates”. *General Relativity and Gravitation* 40.11 (Apr. 2008), pp. 2249–2431. arXiv: 0708.1270 [hep-th].

Implementation in Mathematica

$$\frac{1}{\tilde{\Phi}_{10}(\rho,\sigma,v)} = e^{-2\pi i(\rho+\sigma+v)} \prod_{\substack{k,l,j\in\mathbb{Z},k,l\geq 0\\ j<0 \text{ for } k=l=0}} \left(1-e^{2\pi i(k\sigma+l\rho+jv)}\right)^{-c(kl,j)}$$

We split the calculation into three cases:

1. $k>0$, $l\geq 0$, and $j\in\mathbb{Z}$;
2. $k=0$ but $l>0$, and $j\in\mathbb{Z}$;
3. $k=l=0$ and $j<0$.

We used the following code, with $x=e^{2\pi i\rho}, u=e^{2\pi i\sigma}, w=e^{2\pi iv}$:

```
Φ10reciprocal[x_, u_, w_] :=
 Series[(x u w)^-1 (Product[Series[(1 - x^k u^l w^j)^-c[k,l,j], {x, 0, 5}, {u, 0, 5}], {k, 1, 5}, {l, 0, 5}, {j, -9, 9}])
   (Product[Series[(1 - u^l w^j)^-c[0,l,j], {u, 0, 5}], {l, 1, 5}, {j, -9, 9}])
   (Product[(1 - w^j)^-c[0,0,j], {j, -9, -1}]), {w, ∞, 20}]
result = Φ10reciprocal[x, u, w] // Normal // Expand;
d[qsq_, psq_, qp_] := (-1)^(qp+1) Coefficient[Coefficient[Coefficient[result, x, qsq/2], u, psq/2], w, qp]
```

Black Hole Microstates: Results for K3 $\times T^2$

We obtained the following values for $d(Q, P)$:

$(Q^2, P^2)\backslash Q \cdot P$	-2	0	1	2
$(2, 2)$	–209304	50064	25353	648
$(2, 4)$	–2023536	1127472	561576	50064
$(4, 4)$	–16620544	32861184	18458000	3859456
$(2, 6)$	–15493728	16491600	8533821	1127472
$(4, 6)$	–53249700	632078672	392427528	110910300
$(6, 6)$	2857656828	16193130552	11232685725	4173501828
$(6, 8)$	91631080464	315614079072	233641003920	100673013264

These agree with the values found in the literature.[20]

[20] Ashoke Sen. "How do black holes predict the sign of the Fourier coefficients of Siegel modular forms?" *General Relativity and Gravitation* 43.8 (Apr. 2011), pp. 2171–2183. arXiv: 1008.4209 [hep-th].

Generalisation to Orbifolds of K3 $\times T^2$

The method used here can also be extended to the orbifolds of K3 $\times T^2$.

In particular, we considered the $2A$ orbifold of K3 $\times T^2$.

In that case, the relevant function is a weight-six cusp form $\tilde{\Phi}_6$, and the black hole degeneracies are given by

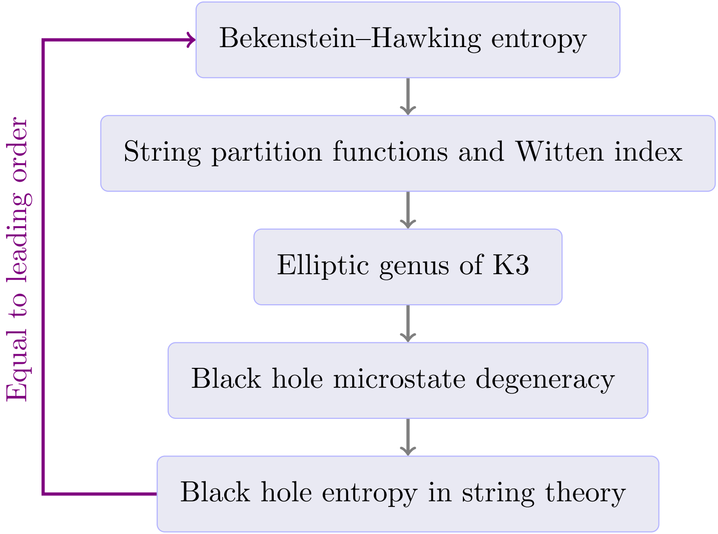
$$d(Q,P) = (-1)^{Q\cdot P+1} g\left(Q^2, \frac{P^2}{4}, Q\cdot P\right),$$

where $g(m,n,p)$ are the Fourier coefficients of $1/\tilde{\Phi}_6$.

Our results again agreed with the values found in the literature.[21]

[21] Ashoke Sen. "How do black holes predict the sign of the Fourier coefficients of Siegel modular forms?" *General Relativity and Gravitation* 43.8 (Apr. 2011), pp. 2171–2183. arXiv: 1008.4209 [hep-th].

Conclusion


Bekenstein–Hawking entropy
String partition functions and Witten index
Elliptic genus of K3
Black hole microstate degeneracy
Black hole entropy in string theory
Equal to leading order

Ramanujan's τ Function

Another important modular form is the modular discriminant, Δ, an example of a modular form of weight 12, defined as

$$\Delta(\tau) := \eta^{24}(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

In 1916, Ramanujan defined the following function, now called the Ramanujan Tau function $\tau(n)$, as the n^{th} Fourier coefficient of Δ.

That is, $\tau(n)$ is defined implicitly as

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n).$$

The first few values are given by

n	1	2	3	4	5	6	7	8
$\tau(n)$	1	-24	252	-1472	4830	-6048	-16744	84480

Lehmer's Conjecture

In 1947, Lehmer[22] conjectured that $\tau(n) \neq 0$. Using advanced methods regarding Galois representations, Derickx, van Hoeij, and Zeng[23] showed that $\tau(n)$ was non-vanishing up to at least $n < 8 \times 10^{23}$.

We considered an extension of this conjecture and considered if the Fourier coefficients of powers of the Δ function are also non-vanishing. Using $\tau_m(n)$ to denote the n^{th} Fourier coefficient of Δ^m, we showed numerically:

Theorem 1

For $1 \leq m \leq 20$ $\tau_m(n)$ is non-vanishing for $n \leq 10^6$.

Theorem 2

For $21 \leq m \leq 100$ $\tau_m(n)$ is non-vanishing for $n \leq 10^5$.

---

[22] D. H. Lehmer. "The vanishing of Ramanujan's function $\tau(n)$". *Duke Mathematical Journal* 14.2 (1947), pp. 429–433.

[23] Maarten Derickx, Mark van Hoeij, and Jinxiang Zeng. "Computing Galois representations and equations for modular curves $X_H(\ell)$". (2013). arXiv: 1312.6819 [math.NT].

Bounds for τ

When defining the τ function Ramanujan made three conjectures[24]. One of these was that for primes p

$$|\tau(p)| \leq 2p^{11/2}$$

We considered the behaviour of integers k such that $|\tau(k)| > 2k^{11/2}$. We will use $k(n)$ to denote the n^{th} value of k for which $|\tau(k)| > 2k^{11/2}$. Using computational methods we found $k(n)$ for $n \leq 83054$.

n	1	2	3	4	5	6	7	8
$k(n)$	799	1751	2987	3149	3713	4841	5321	6157

[24] Srinivasa Ramanujan. "On certain arithmetical functions". *Trans. Cambridge Philos. Soc* 22.9 (1916), pp. 159–184.

Bounds for τ

p	Proportion of $k(n) : p \mid k(n)$	$\frac{1}{p}$
2	0.126243167	0.5
3	0.039384015	0.333333333
5	0.027765069	0.2
7	0.00569509	0.142857143
11	0.067401931	0.090909091
13	0.000854866	0.076923077
17	0.121691911	0.058823529
19	0.034917042	0.052631579
23	0.000734462	0.043478261
29	0.066486864	0.034482759
31	0.000108363	0.032258065
37	2.41×10^{-5}	0.027027027
41	1.20×10^{-5}	0.024390244
43	8.43×10^{-5}	0.023255814
47	0.181400053	0.021276596
53	1.20×10^{-5}	0.018867925
59	0.007501144	0.016949153
61	0.015495942	0.016393443
67	0.068786573	0.014925373

Bounds for τ

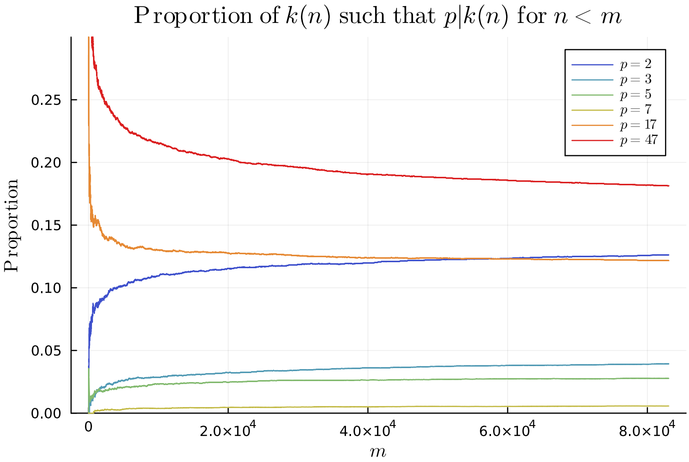


Figure: Plot of the rolling cumulative proportion of $k(n)$ divisible by p for various primes p

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