The symbol alphabet of one-loop Feynman diagrams

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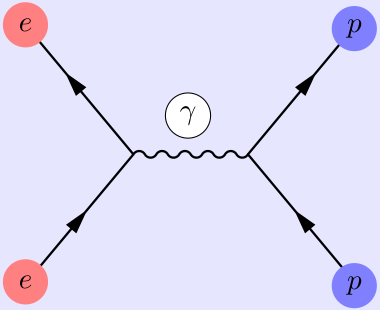
1 Intro & Feynman diagrams

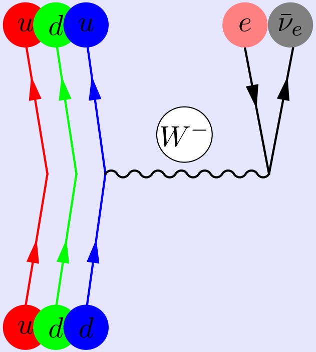
Introduction

Motivation of particle physics: identify what makes up the world, explain physical phenomena from particle interactions (3/4 forces explained by quantum field theory).

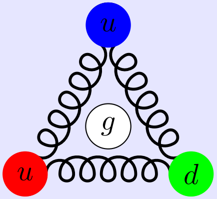
electromagnetic



weak



strong



Introduction

Quantum field theory (**without Feynman diagrams**):

$$\begin{aligned}
\text{Physical system} &\rightarrow \quad \mathcal{L}(\phi, \dot{\phi}, ...) & \text{Lagrangian}\\
\mathcal{L}(\phi, \dot{\phi}, ...) &\rightarrow \quad \mathcal{H}(\phi, \pi, ...) & \text{Hamiltonian}
\end{aligned}$$

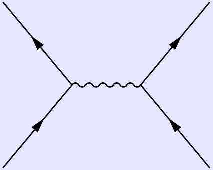
$$\begin{aligned}
\mathcal{H}(\phi, \pi, ...) &\rightarrow \quad \hat{\mathcal{H}}(\hat{\phi}, \hat{\pi}, ...) & \text{Quantisation}\\
\{\phi(\mathbf{x}), \pi(\mathbf{y})\}_{\text{Poisson}} = \delta^{(3)}(\mathbf{x}-\mathbf{y}) &\rightarrow \quad [\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = \mathrm{i}\delta^{(3)}(\mathbf{x}-\mathbf{y}) &\\
\phi(t) &\rightarrow \quad \langle 0|T\{\phi(x)\phi(y)\}|0\rangle &
\end{aligned}$$

$$\begin{aligned}
{}_0\langle \mathbf{k}_1\mathbf{k}_2...|T\{\exp \textstyle\int_{-T}^{T} dt\ H_{\mathrm{I}}(t)\}|\mathbf{p}_1\mathbf{p}_2...\rangle_0 &\rightarrow \quad \langle \mathbf{k}_1\mathbf{k}_2...|\mathcal{S}|\mathbf{p}_1\mathbf{p}_2...\rangle & \text{Scattering}\\
\mathcal{S} &\rightarrow \quad \mathcal{M} &\\
\mathcal{M} &\rightarrow \quad d\sigma/d\Omega, \Gamma, ... &
\end{aligned}$$

Introduction

Quantum field theory (**with Feynman diagrams**):

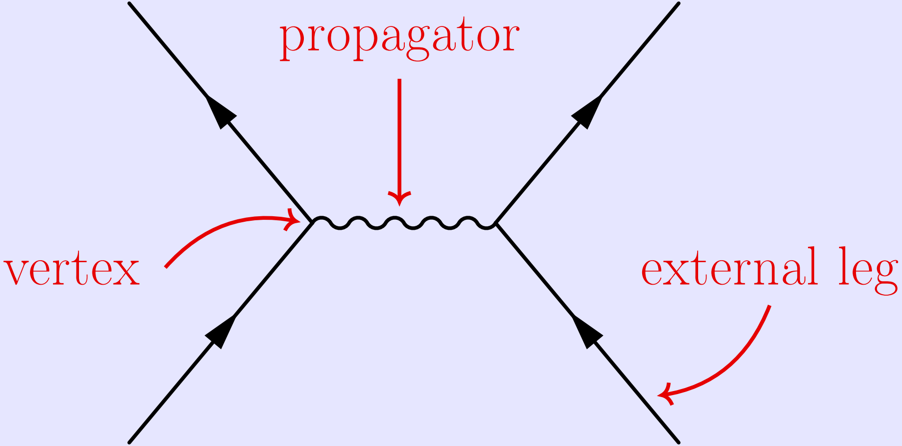
$\mathcal{L} \to$

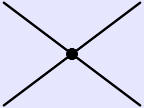
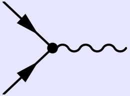
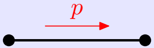
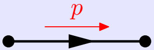
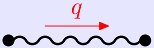
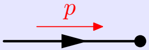
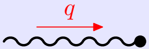


$$\to \mathcal{M} \to \frac{d\sigma}{d\Omega}, \Gamma, \ldots$$

Given Feynman rules, can interpret diagrams to get $\mathcal{M}$. Hooray!

Feynman diagrams



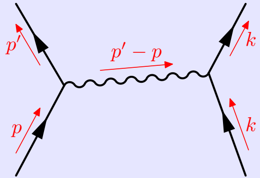
ϕ^4 **theory**	**Quantum ElectroDynamics**
$\mathcal{L}_{\phi^4} = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$	$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^2 + \bar{\Psi}\left(\mathrm{i}\gamma^\mu D_\mu - m\mathbb{1}\right)\Psi$
 $= -\mathrm{i}\lambda$	 $= -\mathrm{i}e\gamma^\mu$
 $= \frac{\mathrm{i}}{p^2 - m^2}$	 $= \frac{\mathrm{i}\left(\gamma^\alpha p_\alpha + m\mathbb{1}\right)}{p^2 - m^2}$
	 $= \frac{-\mathrm{i}\eta_{\mu\nu}}{q^2}$
 $= 1$	 $= u^s(p)$
	 $= \epsilon^\mu(q)$

Feynman diagrams

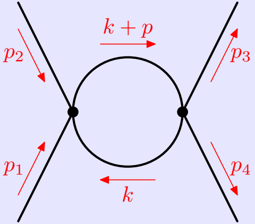
Additional rules:

- Impose momentum conservation at each vertex.
- Integrate over undetermined loop momentum.
- Divide by symmetry factor.

Examples:



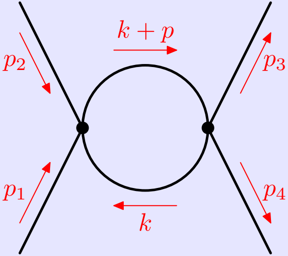
$$\sim \mathrm{i}\mathcal{M} = \bar{u}^{r'}(k')(-\mathrm{i}e\gamma^{\mu})u^{r}(k)\left(\frac{-\mathrm{i}\eta_{\mu\nu}}{(p'-p)^2}\right)\bar{u}^{s'}(p')(-\mathrm{i}e\gamma^{\nu})u^{s}(p),$$



$$\sim \mathrm{i}\mathcal{M} = \frac{1}{2}(-\mathrm{i}\lambda)^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{i}}{(k+p)^2-m^2}\frac{\mathrm{i}}{k^2-m^2}.$$

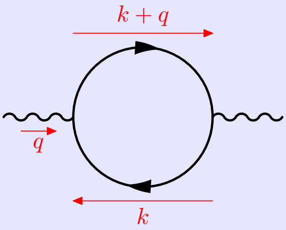
One-loop Feynman integrals

In order to isolate divergences, use dimensional regularisation with $D = 4 - 2\epsilon$:



$$\propto \int d^D k \, \frac{\mathrm{i}}{(k+p)^2 - m^2} \, \frac{\mathrm{i}}{k^2 - m^2} \sim \left[\frac{1}{\epsilon} - \gamma_E - \log\left(\frac{\Delta}{4\pi}\right) + \mathcal{O}(\epsilon)\right]$$

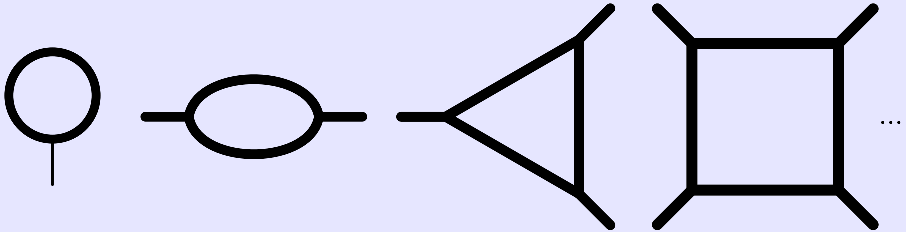
$$\gamma_E = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right)$$



$$\propto \int d^D k \, \frac{\mathrm{tr}\left[\gamma^\mu \left(\not{k} + m\right) \gamma^\nu \left(\not{k} + \not{q} + m\right)\right]}{\left(k^2 - m^2\right)\left((k+q)^2 - m^2\right)} \sim \left(q^2 \eta^{\mu\nu} - q^\mu q^\nu\right) \left[\dots\right]$$

One-loop Feynman integrals

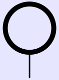
We can express any one-loop Feynman diagram in terms of the first few scalar integrals



which correspond to a basis of integrals called master integrals

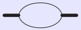
$$I_n^D\left(\{p_i \cdot p_j\}; m_i^2; \epsilon\right) \equiv e^{\gamma_E \epsilon} \int \frac{d^D k}{\mathrm{i}\pi^{D/2}} \frac{1}{(k-p_1)^2 - m_1^2} \frac{1}{(k-p_1-p_2)^2 - m_2^2} \cdots$$

It turns out that all one-loop Feynman integrals are basically logarithms! If $D = 2\left\lceil \frac{n}{2} \right\rceil - 2\epsilon$,



$$= -e^{\gamma_E \epsilon} \left(m^2\right)^{-\epsilon} \Gamma\left(\epsilon\right)$$

$$= -\frac{1}{\epsilon} + \log\left(m^2\right) - \frac{1}{12}\epsilon\left(6\log^2(m^2) + \pi^2\right) + \frac{1}{12}\epsilon^2\left(2\log^3(m^2) + \pi^2\log\left(m^2\right) + 4\zeta(3)\right)$$
$$+ \frac{1}{480}\epsilon^3\left(-160\zeta(3)\log\left(m^2\right) - 20\log^4(m^2) - 20\pi^2\log^2(m^2) - 3\pi^4\right) + O\left(\epsilon^4\right),$$

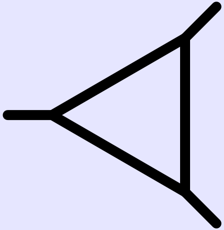


$$= -\frac{2}{\epsilon}\frac{e^{\gamma_E \epsilon}\,\Gamma\left(1-\epsilon\right)^2\Gamma\left(1+\epsilon\right)}{\Gamma\left(1-2\epsilon\right)}(-p^2)^{-1-\epsilon}$$

$$= -\frac{1}{\epsilon} + \log\left(-p^2\right) - \frac{1}{12}\epsilon\left(6\log^2(-p^2) - \pi^2\right) + \frac{1}{12}\epsilon^2\left(2\log^3(-p^2) - \pi^2\log\left(-p^2\right) + 28\zeta(3)\right.$$
$$+ \frac{1}{1440}\epsilon^3\left(-3360\zeta(3)\log\left(-p^2\right) - 60\log^4(-p^2) + 60\pi^2\log^2(-p^2) + 47\pi^4\right) + O\left(\epsilon^4\right).$$

Summary

$$\text{Physics} \sim \mathcal{L} \sim$$



$$\sim \log, \mathrm{Li}_2, \ldots$$

❷ Symbol via coproduct

Symbols: Motivation

The symbol map: A linear map which captures the main combinatorial and analytical properties of certain transcendental functions.

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$$\Big\downarrow \mathcal{S}$$

$$-z \otimes (1-z) + \bar{z} \otimes (1-\bar{z}) - (1-z) \otimes \bar{z} + z \otimes (1-\bar{z}) + (1-\bar{z}) \otimes z - \bar{z} \otimes (1-z)$$

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$$-z\otimes(1-z)+\bar{z}\otimes(1-\bar{z})-(1-z)\otimes\bar{z}+z\otimes(1-\bar{z})+(1-\bar{z})\otimes z-\bar{z}\otimes(1-z)$$

Special cases of the symbol map have been used by mathematicians for over thirty years.

More recently, the symbol map has been introduced into physics, where it can be used to greatly simplify the analytic expressions for many Feynman integrals.

Multiple Polylogarithms (MPLs)

Many Feynman integrals can be expressed in terms of multiple polylogarithms (MPLs).

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Iterated integral representation:

$$G(a_1, a_2, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

Weight of the MPL (n)

In the special case where all a_i are equal to zero,

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Examples

$$G(0; z) = \log z, \qquad G(a; z) = \log\left(1 - \frac{z}{a}\right),$$

$$G(\mathbf{a}_n; z) = \frac{1}{n!} \log^n\left(1 - \frac{z}{a}\right), \quad G(\mathbf{0}_{n-1}, a; z) = -\,\mathrm{Li}_n\left(\frac{z}{a}\right).$$

Coproducts

Let H be a unital associative algebra.

A coproduct is a map $\Delta : H \to H \otimes H$ which is:

1. A homomorphism: $$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$$
2. Coassociative: $$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

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The Coproduct on MPLs

Let $\mathcal{A}$ be the $\mathbb{Q}$-vector space spanned by all MPLs.

Then the quotient space $\mathcal{H} = \mathcal{A}/(i\pi\mathcal{A})$ can be endowed with a coproduct.

The Coproduct on Multiple Polylogarithms

For the ordinary logarithm,

$$\Delta\left(\log(z)\right) = 1 \otimes \log(z) + \log(z) \otimes 1$$

For the classical polylogarithms,

$$\Delta\left(\mathrm{Li}_n(z)\right) = 1 \otimes \mathrm{Li}_n(z) + \sum_{k=0}^{n-1} \frac{1}{k!}\, \mathrm{Li}_{n-k}(z) \otimes \log^k(z)$$

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Examples

$$\Delta\left(\mathrm{Li}_2(z)\right) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \log(1-z) \otimes \log(z)$$

$$\Delta\left(\mathrm{Li}_3(z)\right) = 1 \otimes \mathrm{Li}_3(z) + \mathrm{Li}_3(z) \otimes 1 + \mathrm{Li}_2(z) \otimes \log(z) - \frac{1}{2}\log(1-z) \otimes \log^2(z)$$

(Note that $\mathrm{Li}_1(z) = -\log(1-z)$.)

The Symbol

Coassociativity $\Rightarrow \Delta$ can be uniquely iterated.

The maximal iteration of the coproduct is called the symbol $\mathcal{S}$:

$$\mathcal{S} : \mathcal{H}_n \to \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1$$

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The Symbol Map

To find the symbol of an MPL of weight n:

1. Iteratively apply the coproduct to the MPL n times.
2. Extract the terms in which all entries have weight one (i.e. the ordinary logarithms).

Symbols of MPLs

Some simple examples:

$$\Delta\left(\mathrm{Li}_2(z)\right) = 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 - \log(1-z) \otimes \log z$$

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$$\begin{aligned}(\mathrm{id}\otimes\Delta)\Delta\left(\mathrm{Li}_3(z)\right) &= 1\otimes 1\otimes \mathrm{Li}_3(z) + 1\otimes \mathrm{Li}_3(z)\otimes 1 \\ &+ 1\otimes \mathrm{Li}_2(z)\otimes\log(z) + \mathrm{Li}_2(z)\otimes 1\otimes\log(z) + \mathrm{Li}_2(z)\otimes\log(z)\otimes 1 \\ &- \frac{1}{2} 1\otimes\log(1-z)\otimes\log^2 z - \frac{1}{2}\log(1-z)\otimes 1\otimes\log^2 z \\ &- \frac{1}{2}\log(1-z)\otimes\log^2 z\otimes 1 - \log(1-z)\otimes\log z\otimes\log z\end{aligned}$$

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$$\Rightarrow \mathcal{S}\left(\mathrm{Li}_3(z)\right) = -\log(1-z) \otimes \log z \otimes \log z$$

Symbols of MPLs

Since all entries in the symbol are ordinary logarithms, it is conventional to show only their arguments; for example,

$$\mathcal{S}\left(\mathrm{Li}_2(z)\right) = -(1-z) \otimes z,$$
$$\mathcal{S}\left(\mathrm{Li}_3(z)\right) = -(1-z) \otimes z \otimes z$$

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General Form of the Symbol

In general, the symbol of any MPL f of weight n has the form

$$\mathcal{S}(f) = \sum_{i_1,\dots,i_n} c_{i_1,\dots,i_j} f_{i_1}\otimes\cdots\otimes f_{i_n},$$

where the $c_{i_1,\dots,i_n}$ are coefficients, and the values f_i are known as the symbol letters.

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For Feynman integrals: Letters are functions of the external momenta and propagator masses:

$$f_i = f_i(p_1^2, \dots, p_n^2; m_1^2, \dots, m_k^2).$$

The Symbol Alphabet

The set of all letters which occur in the symbol of a given Feynman integral is known as its symbol alphabet.

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Knowing the alphabet of a given integral is useful because it enables us to write ansätze for the symbol.

$$\text{Alphabet} \rightarrow \text{Symbol} \rightarrow \text{Integral}$$

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Alphabet $\to$ Symbol $\to$ Integral

Our focus: one-loop integrals.

- ☑ **Generic case:** The alphabet for one-loop integrals has been studied in detail for finite integrals where all propagators are massive.
- ☐ **Non-generic case:** The alphabet for non-generic one-loop integrals, such as those involving massless propagators, is not yet fully understood.

❸ Symbol via $\mathcal{A}$-determinant

Symbol of Bubble Diagrams

With a change of variables, the general bubble diagram in $D = 2 - 2\epsilon$ dimensions is given by

$$\widetilde{J}_2\left(p^2; m_1^2, m_2^2\right) = -\frac{e^{\gamma_E \epsilon}\,\Gamma\left(1+\epsilon\right)}{\epsilon}\frac{\left(-p^2\right)^{-1-\epsilon}}{\left(w-\bar{w}\right)^{1+\epsilon}}\left[w^{-\epsilon}{}_2F_1\left(-\epsilon, 1+\epsilon; 1-\epsilon; \frac{w}{w-\bar{w}}\right)\right.$$
$$\left. - \left(w-1\right)^{-\epsilon}{}_2F_1\left(-\epsilon, 1+\epsilon; 1-\epsilon; \frac{w-1}{w-\bar{w}}\right)\right],$$
$$w\bar{w} = \frac{m_1^2}{p^2}, \qquad (1-w)(1-\bar{w}) = \frac{m_2^2}{p^2}, \qquad j_2\left(p^2; m_1^2, m_2^2\right) = -\frac{2}{p^2\left(w-\bar{w}\right)}.$$

We can now evaluate the symbol in order of epsilon.

Symbol of Bubble Diagrams

With a change of variables, the general bubble diagram in $D = 2 - 2\epsilon$ dimensions is given by

$$\widetilde{J}_2\left(p^2; m_1^2, m_2^2\right) = -\frac{e^{\gamma_E \epsilon}\,\Gamma\left(1+\epsilon\right)}{\epsilon}\frac{\left(-p^2\right)^{-1-\epsilon}}{(w-\bar{w})^{1+\epsilon}}\left[w^{-\epsilon}{}_2F_1\left(-\epsilon, 1+\epsilon; 1-\epsilon; \frac{w}{w-\bar{w}}\right)\right.$$
$$\left. - (w-1)^{-\epsilon}{}_2F_1\left(-\epsilon, 1+\epsilon; 1-\epsilon; \frac{w-1}{w-\bar{w}}\right)\right],$$

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$$\Rightarrow \text{the alphabet is } \left\{\frac{1}{2p^2w\bar{w}}, \frac{w-\bar{w}}{1-\bar{w}}, \frac{1-w}{1-\bar{w}}, \frac{w}{1-w}, \frac{\bar{w}}{1-\bar{w}}\right\}.$$

Parameterising Feynman Integrals

Scalar multi-loop Feynman integrals are of the form

$$\mathcal{J} = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n_{int}} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}}$$

and can be parameterised using *Symanzik* graph polynomials $\mathcal{U}$ and $\mathcal{F}$:

$$\mathcal{J}_{\mathrm{F}} = \frac{\Gamma(\nu - \frac{lD}{2})}{\prod_{j=1}^{n_{\mathrm{int}}} \Gamma(\nu_j)} \int_{a_j \geq 0} d^{n_{\mathrm{int}}} a \, \delta(1 - \sum_{j=1}^{n_{\mathrm{int}}} a_j) (\prod_{j=1}^{n_{\mathrm{int}}} a_j^{\nu_j - 1}) \frac{(\mathcal{U})^{\nu-(l+1)D/2}}{(\mathcal{F})^{\nu - lD/2}}$$

.

Computing graph polynomials from spanning trees

$$\mathcal{U} = \sum_{T_i \in \mathcal{T}^1} \prod_{e_i \notin T_i} a_i \qquad \mathcal{F} = \sum_{(T_i, T_j) \in \mathcal{T}^2} \left(\prod_{e_k \notin (T_i, T_j)} a_k \right) \frac{P_{T_i} P_{T_j}}{\mu^2} + \mathcal{U} \sum_i a_i \frac{m_i^2}{\mu^2}.$$

E

Landau Singularities

The quantity

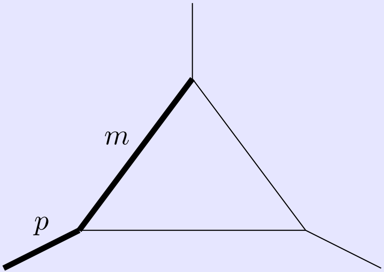
$$\mathcal{Q} = \sum_{e \in E} x_e(-q_e^2 + m_e^2)$$

appears in the denominator during the parameterisation. The Landau equations

$$\begin{cases} \mathcal{Q} = 0 \\ \frac{\partial}{\partial k_l}\mathcal{Q} = 0, \ \forall l = 1, \ldots, L \end{cases}$$

capture type 1 and type 2 singularities associated to the modified Cayley determinant and Gram determinant vanishing respectively.

Triangle graph with massive leg and propagator



$$\mathcal{U} = a_1 + a_2 + a_3,$$
$$\mathcal{F} = -a_3 a_1 p^2 + (a_1 + a_2 + a_3)(a_1 m^2).$$

$$\begin{aligned} \mathcal{J} &= \Gamma(3 - \frac{D}{2}) \int d^3 a \delta(1 - (a_1 + a_2 + a_3)) \frac{\mathcal{U}(a)^{3-D}}{\mathcal{F}(a)^{3-\frac{D}{2}}} \\ &= \frac{\Gamma(1-\epsilon)}{-\epsilon} \int_0^1 da (1-a)^{-\epsilon} (m^2(1 - \frac{p^2}{m^2} a))^{-1-\epsilon} \\ &= \frac{\Gamma(1-\epsilon)}{\epsilon(1-\epsilon)} (m^2)^{-1-\epsilon} \, {}_2F_1(1+\epsilon, 1; 2-\epsilon; \frac{p^2}{m^2}). \end{aligned}$$

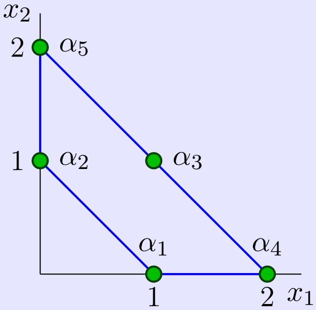
We can always set $\mathcal{U}$ equal to unity for 1 loop.

Bubble Diagram

The Lee-Pomeransky polynomial for the general bubble diagram is given by

$$\mathcal{G} = \mathcal{U} + \mathcal{F} = x_1 + x_2 + (m_1^2 + m_2^2 - p^2)x_1x_2 - m_1^2x_1^2 - m_2^2x_2^2.$$

The convex hull of the exponents of the monomials yields the *Newton polytope*.



$$\begin{aligned}
\Delta_{\alpha_4} &= m_1^2 \\
\Delta_{\alpha_5} &= m_2^2 \\
\Delta_{\alpha_4\alpha_5} &= \lambda(p^2, m_1^2, m_2^2) \\
\Delta_{\alpha_1\alpha_2\alpha_4\alpha_5} &= p^2
\end{aligned}$$

We can write the reduced principal A-determinant as

$$\begin{aligned}
\widetilde{E_A}(\mathcal{G}) &= \Delta_{\alpha_4}\Delta_{\alpha_5}\Delta_{\alpha_4\alpha_5}\Delta_{\alpha_1\alpha_2\alpha_4\alpha_5} \\
&= m_1^2m_2^2(p^4 + m_1^4 + m_2^4 - 2p^2m_1^2 - 2p^2m_2^2 - 2m_1^2m_2^2)p^2\,. \qquad (1)
\end{aligned}$$

Symbol Alphabet for Bubble

In even space-time dimension the letters are

$$W_1 = \frac{-1}{2m_1^2}, \qquad W_2 = \frac{-1}{2m_2^2}, \qquad W_{12} = \frac{2p^2}{\lambda(p^2, m_1^2, m_2^2)},$$

$$W_{(1)2} = \frac{-m_1^2 + m_2^2 + p^2 - \sqrt{\lambda(p^2, m_1^2, m_2^2)}}{-m_1^2 + m_2^2 + p^2 + \sqrt{\lambda(p^2, m_1^2, m_2^2)}},$$

$$W_{1(2)} = \frac{-m_1^2 + m_2^2 - p^2 - \sqrt{\lambda(p^2, m_1^2, m_2^2)}}{-m_1^2 + m_2^2 - p^2 + \sqrt{\lambda(p^2, m_1^2, m_2^2)}}.$$

Open questions

- What is the symbol alphabet?
- At which order ϵ do letters appear?
- Where do letters appear in words relative to each other?
- When can we take limits to recover more diagrams from the general case?

Thank you